PREFACE

This special issue of the Journal of K-Theory is devoted to the subject of motivic homotopy theory introduced by Morel and Voevodsky. Our subject is an outgrowth of techniques due to Serre (sheaf theory), Grothendieck (sites and topologies), and Quillen (model category theory) merged with classical homotopy theory. One of the long-term ambitions of motivic homotopy theory, the solution of the Bloch-Kato conjecture, has recently been achieved in work of Rost, Voevodsky, and others.
Grothendieck's vision of motives in algebraic geometry is largely realized by motivic homotopy theory and motivic cohomology in the sense of Suslin-Voevodsky and Morel. The solution of Milnor's conjecture on Galois cohomology/quadratic forms by Voevodsky and various explicit computations of algebraic K-groups are other major achievements. Motivic homotopy theory has in a relatively short time become a melting pot for interdisciplinary collaborations between mathematicians working in algebra and topology. We mention but two of the current research problems toward which efforts are directed: Morel's work on the Friedlander-Milnor conjecture and the quest for computing motivic stable stems, drawing inspiration from a notoriously difficult problem in topology.
One of the fundamental ideas in motivic homotopy theory is to work with structures more flexible than algebraic varieties considered in classical algebraic geometry, thereby introducing a good notion of homotopy in the algebro-geometric setup. In this setup, one may in some sense tear apart and glue together solution sets of algebraic equations, just as for topological spaces. More precisely, one achieves this by inverting the affine line while taking local data into account. This allows for the promotion of geometric constructions and questions to more subtle forms with arithmetic content.
As demonstrated by the papers in this special volume of the Journal of K-Theory, contemporary research in motivic homotopy theory is driven by methodologies and problems rooted in the fields of algebra and topology. The first two papers and the appendix of the second one focus on applications of motivic homotopy theory. The following two papers study the underlying framework for motivic spaces. The final four contributions concentrate on calculations and tools which were successfully applied in classical homotopy theory and which have analogues in motivic homotopy theory. In the following paragraphs, we give a brief summary of each of these papers.

In the paper "Relatively unramified elements in cycle modules", B. Kahn generalizes work of A. Merkurjev which showed for a smooth projective variety X that the functor which takes a cycle module M to the group of unramified cycle cohomology $A^0(X, M_0)$ is represented by a cycle module K^X given by unramified homology $A_0(X, K_n^M)$. Kahn extends this theorem to all smooth varieties. The representing cycle module is defined in terms of Suslin homology. The reader will find that the methods of proofs in this paper are different from the methods of Merkurjev's.

"Periodicity of Hermitian K-groups" by A. J. Berrick, M. Karoubi and P. A. Østvær shows that hermitian K-theory with finite coefficients satisfies periodicity for a large class of commutative rings which includes rings for which the cup-product with the Bott element is an isomorphism for algebraic K-theory with finite coefficients and rings that are hermitian regular. Moreover the authors extend these results to schemes: they show that hermitian K-theory with finite coefficients and the Bott element inverted is isomorphic to étale hermitian K-theory whenever the cohomological dimension of the scheme is finite.
The appendix by C. Weibel entitled "Bott periodicity for group rings - An appendix to Periodicity in Hermitian K-groups" establishes Bott periodicity for the K-theory with $\mathbb{Z}/m$-coefficients of rings of the form $R[G]$ where R is a ring of S-integers in a number field, G a finite group, and m is prime to the order of G and invertible in R. This isomorphism is given by cup-product with a Bott element and provides the periodicity in algebraic K-theory that is needed to prove periodicity for such group rings in hermitian K-theory.

"Model structures for pro-simplicial presheaves" by J. R. Jardine is devoted to the study of various model structures on the category of pro-objects in simplicial presheaves on an arbitrary small Grothendieck site. The main purpose of these structures is to give a common framework for the comparison of traditional étale homotopy theory and the homotopy theory of simplicial presheaves.

In "Brown representability in $\mathbb{A}^1$-homotopy theory", N. Naumann and M. Spitzweck provide a detailed proof of a Brown representability theorem claimed without proof by V. Voevodsky in his plenary address at the ICM 1998. The proof uses fundamental work of A. Neeman to reduce the problem to one of countability for the compact objects in the stable $\mathbb{A}^1$-homotopy category. We point out to the reader that the main result of this paper is somewhat different in nature to questions considered by J. R. Jardine and others, for the authors consider restrictions to compact objects and show representability of natural transformations.

M. Wendt's paper "Rationally trivial torsors in $\mathbb{A}^1$-homotopy theory" investigates under what conditions do G-torsors for an algebraic group G determine fiber sequences in $\mathbb{A}^1$-homotopy theory. The main result is that if $\pi : E \to X$ is a rationally trivial G-torsor (i.e. trivial over each generic point of X), then π induces a long exact sequence of $\mathbb{A}^1$-homotopy groups. This extends earlier results of F. Morel for SL_n-torsors. Wendt's results apply to a large class of Chevalley groups, and enable him to calculate the second $\mathbb{A}^1$-homotopy group of the projective line.

The last three papers of this collection are devoted to computations in stable motivic homotopy theory. The main tools are versions of the p-completed motivic Adams spectral sequences, first appearing in early work of F. Morel on motivic homotopy theory. These spectral sequences have led to some understanding of the motivic homotopy groups of spheres, and might be successfully applied to investigate classical phenomena of stable homotopy groups of spheres.

In "Convergence of the motivic Adams spectral sequence", P. Hu, I. Kriz, and K. Ormsby discuss convergence of the motivic Adams spectral sequence at a prime p for the stable motivic homotopy category over a field of characteristic zero. Their principal result is that for any cell spectrum X of finite type, this spectral sequence converges to the completion of the homotopy groups of X with respect to p and the Hopf map η. The completion with respect to η is even unnecessary if the virtual cohomological p-dimension of the base field is finite.

In "Motivic invariants of p-adic fields", K. Ormsby specializes the results of the preceding paper in order to analyze the motivic Adams spectral sequences converging to the 2-complete algebraic Johnson-Wilson spectra $BPGL\langle n\rangle$ over p-adic fields. In particular, over p-adic fields, the 2-complete $BPGL\langle n\rangle$ splits over 2-complete $BPGL\langle 0\rangle$ which implies that the slice spectral sequence for $BPGL$ collapses. This paper reveals interesting phenomena which are special to the case of p-adic fields.

The paper "Motivic connective K-theories and the cohomology of $A(1)$" by D. C. Isaksen and A. Shkembi provides interesting computations in 2-completed stable motivic homotopy theory over $\mathbb{C}$. In classical theory, the calculations involving the Hopf map η are based on real K-theory and stable connective real K-theory. The authors construct motivic analogues of stable connective real and complex K-theory and compute the E_2-terms of the associated motivic Adams spectral sequence at the prime 2.

The editors (Friedlander, Østvær, Quick) organized a Workshop on Motivic Homotopy Theory at the University of Münster in July 2009. The articles comprising this special issue are authored by participants of this workshop and their collaborators.

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